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Composition operators induced by universal covering maps

Joint International Meeting AMS, EMS, and SPM

Matthew M. Jones

Middlesex University London

13th June 2015

Preliminaries

- $\mathbb{D} = \{z: |z| < 1\}$
- H^p ($1 \leq p < \infty$) the Hardy space, $f \in H^p$ if and only if

$$\|f\|_p^p = \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty$$

- If $\phi: \mathbb{D} \rightarrow \mathbb{D}$ is holomorphic then the **composition operator** C_ϕ is

$$\begin{aligned} C_\phi &: H^p \rightarrow H^p \\ f &\mapsto f \circ \phi \end{aligned}$$

Theorem (Littlewood's subordination theorem (1925))

Suppose that $\phi: \mathbb{D} \rightarrow \mathbb{D}$ is univalent and holomorphic, $\phi(0) = 0$. If $f \in H^p$ ($1 \leq p < \infty$) then

$$\int_0^{2\pi} |f(\phi(re^{i\theta}))|^p d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^p d\theta$$

Boundedness of C_ϕ

Theorem (Littlewood's subordination theorem (1925))

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$$\int_0^{2\pi} |f(\phi(re^{i\theta}))|^p d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^p d\theta$$

So

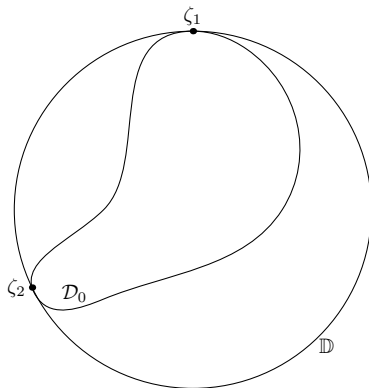
$$C_\phi: H^p \rightarrow H^p$$

is a bounded operator.

Compactness of C_ϕ

If $\phi: \mathbb{D} \rightarrow \mathcal{D}_0$ is univalent then C_ϕ is compact if and only if

$$\lim_{|z| \rightarrow 1} \frac{1 - |\phi(z)|}{1 - |z|} = \infty.$$

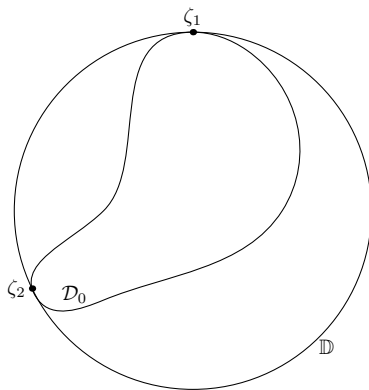


Compactness of C_ϕ

If $\phi: \mathbb{D} \rightarrow \mathcal{D}_0$ is univalent then C_ϕ is compact if and only if

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The **angular derivative** does not exist anywhere.

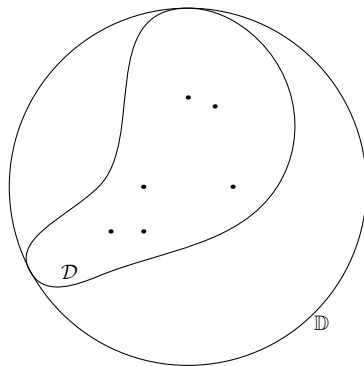


Compactness of C_ϕ and $\phi(\mathbb{D})$

Consider instead domains of the form

$$\mathcal{D} = \mathcal{D}_0 \setminus \{p_1, p_2, \dots, p_n\}$$

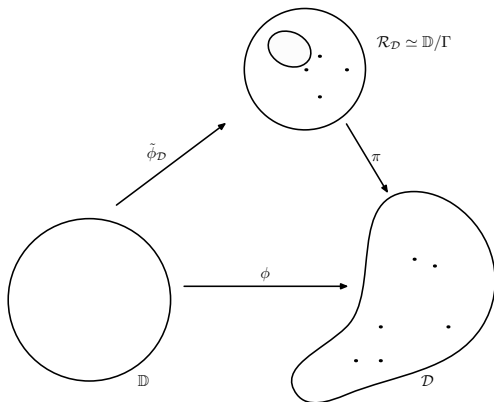
where p_k , $k = 1, 2, \dots, n$ are isolated points in \mathcal{D}_0 .



The uniformization theorem

- For simply connected \mathcal{D}_0 we have the Riemann mapping theorem
- There is a 'unique' univalent mapping ψ of \mathbb{D} onto \mathcal{D}_0 .
- For domains $\mathcal{D} = \mathcal{D}_0 \setminus \{p_1, p_2, \dots, p_n\}$ we must employ the uniformization theorem
- There is a holomorphic **universal covering map** of \mathbb{D} onto \mathcal{D} .

The universal covering map



$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\tilde{\phi}_D} & \mathcal{R}_D \\ & \searrow \phi & \downarrow \pi \\ & & \mathcal{D} \end{array}$$

Example

- Riemann mapping from \mathbb{D} to \mathbb{D} is $\psi(z) = z$
- A universal covering map from \mathbb{D} to $\mathbb{D} \setminus \{0\}$ is

$$\phi(z) = \exp\left(-\frac{1+z}{1-z}\right)$$

- Note ϕ is an inner function with

$$\lim_{r \rightarrow 1^-} |\phi(re^{i\theta})| = \begin{cases} 0, & \theta = 0 \\ 1, & \text{otherwise} \end{cases}$$

Multivalent functions

Definition (Nevanlinna's counting function)

For any function $\phi: \mathbb{D} \rightarrow \mathbb{D}$

$$\mathcal{N}_{\phi}(w) = \begin{cases} \sum_{z: \phi(z)=w} \log \frac{1}{|z|} & w \in \phi(\mathbb{D}) \\ 0 & w \in \mathbb{D} \setminus \overline{\phi(\mathbb{D})} \end{cases}$$

Multivalent functions

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Theorem (Shapiro's compactness criterion)

C_ϕ is compact on H^p if and only if

$$\lim_{|w| \rightarrow 1} \frac{\mathcal{N}_\phi(w)}{\log 1/|w|} = 0$$

Proof of results

For our universal covering map

$$\phi: \mathbb{D} \rightarrow \mathcal{D} = \mathcal{D}_0 \setminus \{p_1, \dots, p_n\}$$

we have:

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\tilde{\phi}_{\mathcal{D}}} & \mathcal{R}_{\mathcal{D}} \\ & \searrow \phi & \downarrow \pi \\ & & \mathcal{D} \end{array}$$

- $\mathcal{R}_{\mathcal{D}} \simeq \mathbb{D}/\Gamma$ for Γ a torsion-free Fuchsian group
- $\Lambda(\Gamma)$ – the limit set of Γ – satisfies $\Lambda(\Gamma) \subsetneq \partial\mathbb{D}$
- $\phi^{-1}(w)$ is a Γ -orbit
- $z_1, z_2 \in \phi^{-1}(w)$ if and only if $\exists h \in \Gamma$ such that $z_1 = h(z_2)$.

Main results

Theorem

Suppose that $\mathcal{D} = \mathcal{D}_0 \setminus \{p_1, \dots, p_n\}$ and ϕ is a universal covering of \mathbb{D} onto \mathcal{D} . Then C_ϕ is compact on H^p , $1 \leq p < \infty$, if and only if

$$\lim_{z \rightarrow \zeta} \frac{1 - |\phi(z)|}{1 - |z|} = \infty$$

for all $\zeta \in \partial\mathbb{D}$.

Main results

Theorem

Suppose that $\mathcal{D} = \mathcal{D}_0 \setminus \{p_1, \dots, p_n\}$ and ϕ is a universal covering of \mathbb{D} onto \mathcal{D} . Then C_ϕ is compact on H^p , $1 \leq p < \infty$, if and only if

$$\lim_{z \rightarrow \zeta} \frac{1 - |\phi(z)|}{1 - |z|} = \infty$$

for all $\zeta \in \partial\mathbb{D}$.

Theorem

Suppose that $\mathcal{D} = \mathcal{D}_0 \setminus \{p_1, \dots, p_n\}$, ϕ is a universal covering of \mathbb{D} onto \mathcal{D} , and ψ is the univalent Riemann mapping of \mathbb{D} onto \mathcal{D}_0 . Then C_ϕ is compact on H^p , $1 \leq p < \infty$, if and only if C_ψ is.

Definition (The Poincare series)

For a Fuchsian group Γ and $z, w \in \mathbb{D}$

$$\rho_{\Gamma}(z, w; s) = \sum_{g \in \Gamma} \exp -s d_{\mathbb{D}}(z, g(w))$$

where

$$d_{\mathbb{D}}(z, w)$$

is the hyperbolic distance in \mathbb{D} .

Lemma

There are constants c_1 and c_2 such that for z with $w = \phi(z)$ suitably chosen

$$c_1 \rho_{\Gamma}(0, z; 1) \leq \mathcal{N}_{\phi}(w) \leq c_2 \rho_{\Gamma}(0, z; 1)$$

Proof of result

Lemma

C_ϕ is compact on H^p if and only if for all $\zeta \notin \Lambda(\Gamma)$

$$\lim_{z \rightarrow \zeta} \frac{\rho_\Gamma(0, z; 1)}{1 - |\phi(z)|} = 0$$

Lemma

If Γ is finitely generated then there are constants c_1 and c_2 such that for z close enough to $\partial\mathbb{D} \setminus \Lambda(\Gamma)$

$$c_1(1 - |z|^2) \leq \rho_\Gamma(0, z; 1) \leq c_2(1 - |z|^2)$$

Proof of main result

- ① C_ϕ compact if and only if for all $\zeta \notin \Lambda(\Gamma)$

$$\lim_{z \rightarrow \zeta} \frac{\rho_\Gamma(0, z; 1)}{1 - |\phi(z)|} = \lim_{z \rightarrow \zeta} \frac{1 - |z|}{1 - |\phi(z)|} = 0$$

- ② $\Lambda(\Gamma)$ consists of:

- fixed points of parabolic elements (correspond to p_i)
- points of approximation (where orbits converge non-tangentially and therefore ϕ is fixed on a sequence converging non-tangentially)

Extension of work

- These results can be extended to general (finitely) multiply connected domains
- Widely applicable in the study of composition operators since the Nevanlinna counting function appears often
- The characterisation of Schatten class composition operators in particular employs \mathcal{N}_ϕ .